Lattice Gauge Fixing and Gribov Copies on the Lattice

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Flavor physics

- Indirect CP violation: ϵ_K
  - Hadronic matrix element: B_K and $B_2, \cdots, B_5$ [Jaehoon Leem]
  - CKM matrix element: V_{cb} [Yong-Chull Jang]
- Lattice QCD calculation as a high precision test of the standard model

Renormalization

- Hadronic matrix element: B_K

$$O_{\Delta S=2} = \sum_{\nu} [\bar{s}\gamma_\nu(1-\gamma_5)d][\bar{s}\gamma_\nu(1-\gamma_5)d] \tag{1}$$

- Renormalization
  - Lattice perturbation theory [J.J.Kim et al. PRD 81 (2010) 114503, PRD 83 (2011) 094503]
  - Non-perturbative renormalization (NPR) [J.H.Kim et al. arXiv:1410.6607]

Non-perturbative renormalization (NPR)

[G.Martinelli et al. NPB 445 (1995) 81-105]

- Regularization independent (RI or RI-MOM) scheme
  - renormalization condition on a correlation function
  - with fixed external momenta
- Correlation function with external quark state
  - need to fix the gauge

Landau gauge fixing

From the initial set of gauge configuration, we numerically obtain a new set by gauge transformation

$$\{A_\mu(x)\}, \{U_\mu(x)\} \qquad \rightarrow \qquad \{A_\mu^g(x)\}, \{U_\mu^g(x)\}, \tag{2}$$

such that

$$\partial_\mu A_\mu^g(x) = 0. \tag{3}$$

Minimizing functional

In Abelian gauge theory on continuum, minimizing the positive definite functional

$$F = \int d^4x \; A_\mu(x) A_\mu(x) \tag{4}$$

gives a gauge fixing. Under the gauge transformation $A_\mu \to A_\mu - \partial_\mu \chi$,

$$\delta F = 2 \int d^4x \; \chi(x) \partial_\mu A_\mu(x), \tag{5}$$

we have the Landau gauge fixing condition

$$\delta F = 0 \quad \leftrightarrow \quad \partial \cdot A(x) = 0. \tag{6}$$

Non-Abelian gauge theory

In non-Abelian gauge theory, we have an analogous functional

$$F = \int d^4x \; \mathrm{tr}[A_\mu(x)A_\mu(x)]. \tag{7}$$

The gauge transformation with $G(x) = e^{i\omega(x)} \in SU(N_C)$ is

$$A_\mu(x) \to G(x)A_\mu(x)G(x)^\dagger + i(\partial_\mu G(x))G(x)^\dagger \tag{8}$$

Under this variation,

$$\delta F = 2\int d^4x \; \mathrm{tr}[(\partial \cdot A)\omega], \tag{9}$$

the extrimizing condition $\delta F = 0$ for an arbitrary variation $\omega(x)$ requires the Landau gauge condition, $\partial \cdot A(x) = 0$.

Lattice gauge theory (1)

On the lattice, we write the functional in terms of link variable $U_\mu(x)$,

$$F_L = \sum_{\mu,x} \mathrm{Tr}[U_\mu(x) + U_\mu(x)^\dagger]. \tag{10}$$

By using the expression $U_\mu(x) = e^{iaA_\mu(x)} \in SU(N_C)$ where a is the lattice spacing, lattice functional is equivalent to the previous functional in the continuum limit ($a \to 0$),

$$F_L = \sum_{\mu,x} \mathrm{Tr}[2 - a^2 A_\mu(x) A_\mu(x) + O(a^3)]. \tag{11}$$

Lattice gauge theory (2)

The gauge transformation of the link with $G(x) = e^{i\omega(x)} \in SU(N_C)$ is

$$U_\mu(x) \to G(x) U_\mu(x) G^\dagger(x+\mu). \tag{12}$$

By some algebra, the functional variation with respect to $\omega(x) = \omega^a(x) T^a$ is

$$\begin{aligned} \frac{\delta F_L}{\delta \omega^a(x)} T^a &= -\frac{i}{2} \sum_\mu \delta_{-\mu} \left[[U_\mu(x) - U_\mu(x)^\dagger] - \frac{1}{N_C} \mathrm{Tr}[U_\mu(x) - U_\mu(x)^\dagger] \right] \\ &\equiv -\frac{i}{2} \Delta(x) \end{aligned} \tag{13}$$

where $a = 1$, and $\delta_\mu f(x) = f(x+\mu) - f(x)$. Lattice version of the Landau gauge is $\Delta(x) = 0$.

Lattice gauge theory (3)

$\{U_\mu(x)\}$: gauge configuration, a set of gauge links.

$$\Delta(x) = \sum_\mu \delta_{-\mu} \left[[U_\mu(x) - U_\mu(x)^\dagger] - \frac{1}{N_C} \mathrm{Tr}[U_\mu(x) - U_\mu(x)^\dagger] \right] \tag{14}$$

$$= -2ia^2 \partial \cdot A \quad (a \to 0 \text{ limit}) \tag{15}$$

Landau gauge fixing condition on the lattice

$$\Delta(x) = 0 \quad \forall x. \tag{16}$$

$$\text{or,} \quad \theta \equiv \sum_x \mathrm{Tr}[\Delta(x)\Delta^\dagger(x)] = 0 \tag{17}$$

Method of the steepest descent

Consider a function $f(x^a)$ which has a maximum and its variation

$$f(x + \delta x) = f(x) + \delta x^a \nabla^a f(x) + O(\delta x^2). \tag{18}$$

Note that the vector $\nabla^a f(x)$ indicates the direction of steepest ascent. By choosing the variation along the direction

$$\delta x^a = \alpha \nabla^a f(x), \tag{19}$$

with small step size $\alpha > 0$, the function clearly increases,

$$f(x) > f(x + \delta x) = f(x) + \alpha(\nabla f(x))^2 + O(\alpha^2). \tag{20}$$

Method of the steepest descent (2)

[C.T.H.Davies et al, PRD 37, 1581 (1988)]
Similarly, for the functional F_L of lattice Landau gauge fixing,

$$F_L^g = F_L + \sum_x \omega^a(x) \frac{\delta F_L}{\delta \omega^a(x)} + O(\omega^2) \tag{21}$$

The gauge tranformation with the method of steepest descent is

$$\omega(x) = \alpha \frac{\delta F_L}{\delta \omega^a(x)} T^a = -\frac{i\alpha}{2}\Delta(x), \tag{22}$$

$$\text{or, } G(x) = \exp(\frac{\alpha}{2}\Delta(x)) \tag{23}$$

where $\Delta(x)$ is calculated from the gauge configuration $\{U_\mu(x)\}$ (: a set of gauge link).

Algorithm

- Choose small α.
- *i-th iteration*:
  Starts from the gauge configuration $\{U_\mu^{(i)}(x)\}$, calculate $\Delta^{(i)}(x)$ and

$$G^{(i)}(x) = \exp(\frac{\alpha}{2}\Delta^{(i)}(x)), \tag{24}$$

  and update the configuration

$$U_\mu^{(i+1)}(x) = G^{(i)}(x)U_\mu^{(i)}(x)G^{(i)}(x+\hat{\mu})^\dagger. \tag{25}$$

  Calculate and see the value of $\Delta^{(i+1)}(x)$ or $\theta^{(i+1)}$.

Fourier acceleration

In the continuum QCD,

$$\Delta^{(i)} \to \partial \cdot A^{(i)} = \partial \cdot A^{(i-1)} + \alpha(\partial_\nu D_\nu)\partial \cdot A^{(i-1)} \tag{26}$$

where $D_\nu f = \partial_\nu f - i[A_\nu, f]$. In abelian limit, each Fourier component decays as follows,

$$\partial \cdot A^{(i)}(p) = \underbrace{(1 - \alpha p^2)}_{\approx exp[-\alpha p^2]} \partial \cdot A^{(i-1)}(p). \tag{27}$$

Acceleration is,

$$\alpha \to \alpha \frac{p^2_{max}}{p^2}. \tag{28}$$

Effect of the acceleration

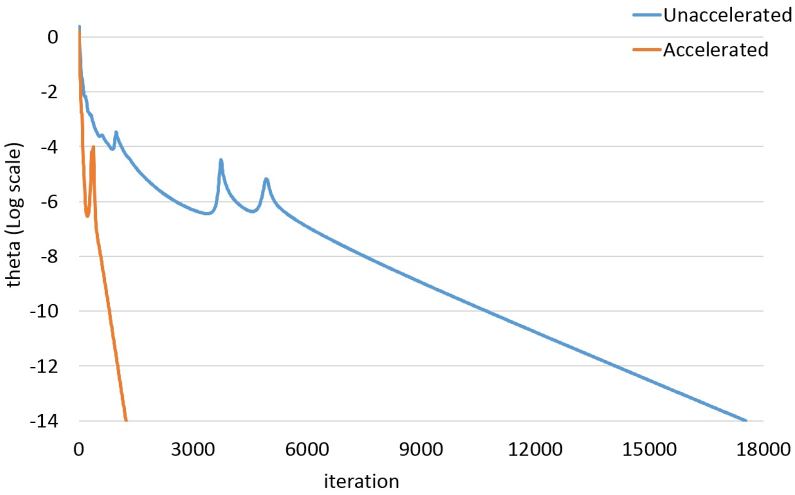


Figure : Iteration on $8^3 \times 32$ lattice

Comparing various algorithms

Tested with node geometry (2,2,1,1)

Algorithm	iter	time(s)
Cabbibo-Marinari	7200	166
SD	17530	304
SDFA (FFT)	1250	31

Table : $8^3 \times 32$ lattice configuration

Volume dependence of iteration time

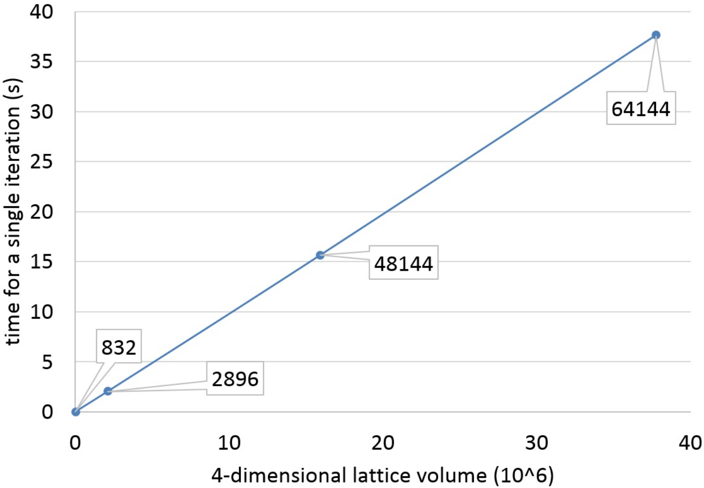


Figure : Single iteration time of various lattice volume with node geometry (2,2,2,2)

Gribov copy

- Gribov (1978) discovered that for non-abelian gauge theories, usual linear gauge conditions does not fix the gauge fields in a unique way.
- There can be two different configurations that both satisfy the gauge fixing condition, but related by nontrivial gauge tr. Simply,

$$\{U\} \to \{U^{g1}\}, \{U^{g2}\} \tag{29}$$

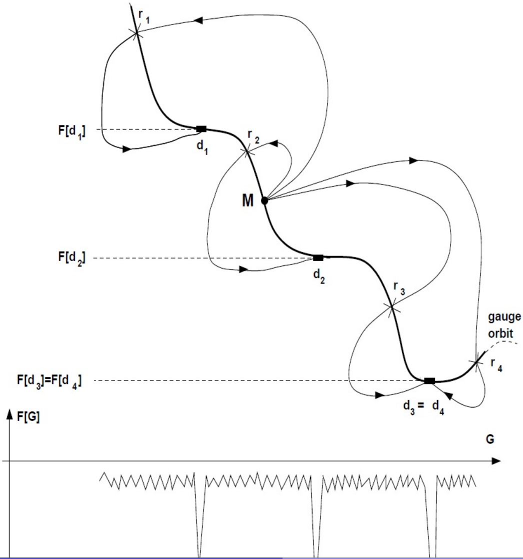
such that $\theta[U^{g1}] = \theta[U^{g2}] = 0$, but $\{U^{g1}\} \nsim \{U^{g2}\}$.

Gribov uncertainty

Matrix elements between quark states

- Need gauge fixing.
- Additional Gribov copy degree exists and that may appear in the result with statistical uncertainty.

Generation of Gribov copy



Schematic cartoon of Gribov copy generation with random gauge transformation [L. Giusti et al, Int. J. Mod. Phys. A 16, 3487 (2001)]

The landscape of minima of the functional F_L

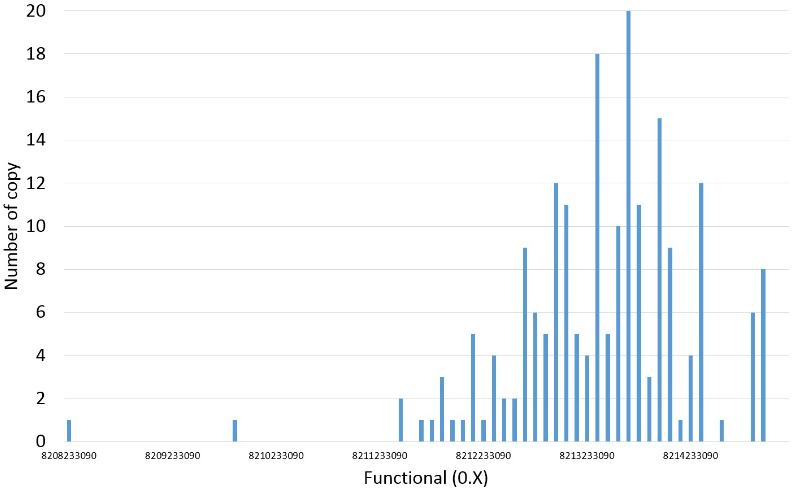


Figure : Histogram of 200 confs. generated from a single $8^3 \times 32$, $\beta = 5.7$ conf.

Future plan

- Multi-GPU implementation of the gauge fixing algorithm
- Gribov copy dependence of NPR related to the neutral Kaon mixing